Knot-Logic and Majorana Fermions
L. H. Kauffman, UIC
www.math.uic.edu/~kauffman
<kauffman@uic.edu>
Discrimination, Process, Symmetry, Knotlogic
Quantum Mechanics in a Nutshell

0. A state of a physical system corresponds to a unit vector $|S\rangle$ in a complex vector space.

1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector: $|S\rangle \longrightarrow U|S\rangle$

2. If $|S\rangle = z_1|e_1\rangle + z_2|e_2\rangle + \ldots + z_n|e_n\rangle$ in a measurement basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$, then measurement of $|S\rangle$ yields $|e_i\rangle$ with probability $|z_i|^2$. 
Recalling the Diffusion Equation

\[ P(x,t) = \text{Probability that the particle is at } x \text{ at time } t. \]

\[ P(x,t+dt) = \frac{1}{2}[P(x+dx,t) + P(x-dx,t)] \]

Therefore,

\[ P(x,t+dt) - P(x,t) = \frac{1}{2}[P(x+dx,t) - P(x,t) - (P(x,t) - P(x-dx,t))] \]

\[ \frac{dP}{dt} = \left(\frac{K}{2}\right) \frac{d^2P}{dx^2} \quad \text{Diffusion Equation} \]

\[ K = \left(\frac{dx}{dt}\right)^2 \quad \text{Diffusion Constant} \]
An Imaginary Tale

God Does Not Play Dice! Here is a little story about the square root of minus one and quantum mechanics. God said - I would really like to be able to base the universe on the Diffusion Equation

$$\frac{\partial \psi}{\partial t} = \kappa \frac{\partial^2 \psi}{\partial x^2}.$$ 

But I need to have some possibility for interference and waveforms. And it should be simple. So I will just put a “plus or minus” ambiguity into this equation, like so:

$$\pm \frac{\partial \psi}{\partial t} = \kappa \frac{\partial^2 \psi}{\partial x^2}.$$ 

$$\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi.$$ 

Schrödinger
This is good, but it is not quite right. I do not play dice. The $\pm$ coefficient will have to be lawful, not random. Nothing is random. What to do? I shall take $\pm$ to mean the alternating sequence

$$\pm = \cdots + - + - + - + - + \cdots$$

and time will become discrete. Then the equation will become a difference equation in space and time

$$\psi_{t+1} - \psi_t = (-1)^t \kappa(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx))$$

where

$$\partial_x^2 \psi_t = \psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx).$$

This will do it, but I have to consider the continuum limit. But there is no meaning to

$$(-1)^t$$
in the realm of continuous time. What do do? In the discrete world my wave function (not a bad name for it!) divides into $\psi_e$ and $\psi_o$ where the time is either even or odd. So I can write

$$\partial_t \psi_e = \kappa \partial_x^2 \psi_o$$

$$\partial_t \psi_o = -\kappa \partial_x^2 \psi_e.$$  

I will take the continuum limit of $\psi_e$ and $\psi_o$ separately!
I shall call this the Schroedinger equation. Instead of the simple diffusion equation, I have a mutual dependency where the temporal variation of $\psi_e$ is mediated by the spatial variation of $\psi_o$ and vice-versa. This is the price I pay for not playing dice.

$$\psi = \psi_e + i\psi_o$$

$$\partial_t \psi_e = \kappa \partial^2_x \psi_o$$

$$\partial_t \psi_o = -\kappa \partial^2_x \psi_e.$$ 

$$i\partial \psi / \partial t = \kappa \partial^2 \psi / \partial x^2.$$
Finally, a use for that so called imaginary number that Merlin has been bothering me with (You might wonder how Merlin could do this when I have not created him yet, but after all I am that am.). This $i$ has the property that $i^2 = -1$ so that

$$i(A + iB) = iA - B$$

when $A$ and $B$ are ordinary numbers,

$$i = -1/i,$$

and so you see that if $i = 1$ then $i = -1$, and if $i = -1$ then $i = 1$. So $i$ just spends its time oscillating between $+1$ and $-1$, but it does it lawfully and so I can regard it as a definition that

$$i = \pm 1.$$
In fact, I can see now what Merlin was getting at. When I multiply \( ii = (\pm 1)(\pm 1) \), I get \(-1\) because the \( i \) takes a little time to oscillate and so by the time this second term multiplies the first term, they are just out of phase and so we get either \((+1)(-1) = -1\) or \((-1)(+1) = -1\). Either way, \( ii = -1 \) and we have the perfect ambiguity.
But now with this Schroedinger equation, I should write

$$\psi = A + Bi$$

and then, since $i(A + Bi) = -B + Ai$, the new equation becomes two equations

$$\frac{\partial A}{\partial t} = \kappa \frac{\partial^2 B}{\partial x^2},$$

and

$$-\frac{\partial B}{\partial t} = \kappa \frac{\partial^2 A}{\partial x^2}.$$  

Instead of the simple diffusion equation, I have a mutual dependency where the temporal variation of $A$ is mediated by the spatial variation of $B$ and vice-versa. This is the price I pay for not playing dice.
Question: Is there a relativistic motivation for this binary clock image of the Schrodinger equation? (Compare with Garnet Ord)
\[
\frac{\psi_{x+1} - \psi_x}{dt} = (-1)^x \left[ \frac{(dx)^2}{2dt} \right] \left[ \psi_x(x+dx) - 2\psi_x(x) + \psi_x(x-dx) \right]
\]

\[
\psi_x(x) = \frac{(-1)^x}{2} \left[ \psi_x(x-dx) + \psi_x(x+dx) \right] + (1-(-1)^x)\psi_x(x)
\]

The Schrödinger Equation solutions are limits of a discrete process with implicit parity.
THE SQUARE ROOT OF MINUS ONE IS A CLOCK.

From $G = \sqrt{G}$ to $i = -1/i$.

$i$ as an imaginary value, defined in terms of itself.

$$i = -1/i$$

$$ii = -1$$

The square root of minus one “is” a discrete oscillation.

$$... +1, -1, +1, -1, +1, -1, ...$$

$[-1,+1]$ $[+1,-1]$
We introduce a *temporal shift operator* $\eta$ such that

$$[a, b] \eta = \eta [b, a]$$

and

$$\eta \eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\cdots abababab \cdots .$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define $i$ by the equation

$$i = [1, -1] \eta.$$

This makes $i$ both a value and an operator that takes into account a step in time.

We calculate

$$ii = [1, -1] \eta [1, -1] \eta = [1, -1][-1, 1] \eta \eta = [-1, -1] = -1.$$
\[ e = [1, -1]. \]

\[ e^2 = [1, 1] = 1 \]

\[ [a, b] \eta = \eta [b, a] \quad i = [1, -1] \eta. \]

\[ e^2 = 1, \]

\[ \eta^2 = 1, \]

\[ e \eta = -\eta e. \]

\[ i i = [1, -1] \eta [1, -1] \eta = [1, -1] [-1, 1] \eta \eta = [-1, -1] = -1. \]
Let $A = [a,b]$ and $B = [c,d]$ and let $C = [r,s]$, $D = [t,u]$. With $A' = [b,a]$, we have

$$(A + B\eta)(C+D\eta) = (AC + BD') + (AD + BC')\eta.$$  

This writes $2 \times 2$ matrix algebra in the form of a hypercomplex number system. From the point of view of iterants, the sum $[a,b] + [b,c]\eta$ can be regarded as a superposition of two types of observation of the iterants $I\{a,b\}$ and $I\{c,d\}$. The operator-view $[c,d]\eta$ includes the shift that will move the viewpoint from $[c,d]$ to $[d,c]$, while $[a,b]$ does not contain this shift. Thus a shift
5.2 Relativity and the Dirac Equation

Starting with the algebra structure of $e$ and $\eta$ and adding a commuting square root of $-1$, $i$, we have constructed fermion algebra and quaternion algebra. We can now go further and construct the Dirac equation. This may sound circular, in that the fermions arise from solving the Dirac equation, but in fact the algebra underlying this equation has the same properties as the creation and annihilation algebra for fermions, so it is by way of this algebra that we will come to the Dirac equation. If the speed of light is equal to 1 (by convention), then energy $E$, momentum $p$ and mass $m$ are related by the (Einstein) equation

$$E^2 = p^2 + m^2.$$
\[ \hat{E} = i \frac{\partial}{\partial t} \]

\[ \hat{p} = -i \frac{\partial}{\partial x} \]

\[ \hat{E}\psi = E\psi \]

\[ \hat{p}\psi = p\psi. \]
Dirac constructed his equation by looking for an algebraic square root of $p^2 + m^2$ so that he could have a linear operator for $E$ that would take the same role as the Hamiltonian in the Schrödinger equation. We will get to this operator by first taking the case where $p$ is a scalar (we use one dimension of space and one dimension of time. Let $E = \alpha p + \beta m$ where $\alpha$ and $\beta$ are elements of a a possibly non-commutative, associative algebra. Then

$$E^2 = \alpha^2 p^2 + \beta^2 m^2 + pm(\alpha \beta + \beta \alpha).$$

Hence we will satisfy $E^2 = p^2 + m^2$ if $\alpha^2 = \beta^2 = 1$ and $\alpha \beta + \beta \alpha = 0$. This is our familiar Clifford algebra pattern and we can use the iterant algebra generated by $e$ and $\eta$ if we wish. Then, because the quantum operator for momentum is $-i\partial/\partial x$ and the operator for energy is $i\partial/\partial t$, we have the Dirac equation

$$i\partial\psi/\partial t = -i\alpha\partial\psi/\partial x + \beta m\psi.$$ 

Let

$$\mathcal{O} = i\partial/\partial t + i\alpha\partial/\partial x - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O}\psi(x, t) = 0.$$
$$\mathcal{O} = i \partial/\partial t + i\alpha \partial/\partial x - \beta m$$

Now note that

$$\mathcal{O} e^{i(px-Et)} = (E - \alpha p + \beta m) e^{i(px-Et)}$$

and that if

$$U = (E - \alpha p + \beta m) \beta \alpha = \beta \alpha E + \beta p + \alpha m,$$

then

$$U^2 = -E^2 + p^2 + m^2 = 0,$$

from which it follows that

$$\psi = U e^{i(px-Et)}$$

is a (plane wave) solution to the Dirac equation.

In fact, this calculation suggests that we should multiply the operator $\mathcal{O}$ by $\beta \alpha$ on the right, obtaining the operator

$$\mathcal{D} = \mathcal{O} \beta \alpha = i\beta \alpha \partial/\partial t + i\beta \partial/\partial x + \alpha m,$$ 

and the equivalent Dirac equation

$$\mathcal{D} \psi = 0.$$ 

In fact for the specific $\psi$ above we will now have $\mathcal{D}(U e^{i(px-Et)}) = U^2 e^{i(px-Et)} = 0$. This way of reconfiguring the Dirac equation in relation to nilpotent algebra elements $U$ is due to Peter Rowlands [94]. We will explore this relationship with the Rowlands formulation in a separate paper.
If we let
\[ \tilde{\psi} = e^{i(px + Et)} \]
(reversing time), then we have
\[ \mathcal{D}\tilde{\psi} = (-\beta \alpha E + \beta p - \alpha m)\psi = U^\dagger\tilde{\psi}, \]
giving a definition of $U^\dagger$ corresponding to the anti-particle for $U\psi$.

We have that
\[ U^2 = U^\dagger^2 = 0 \]
and
\[ UU^\dagger + U^\dagger U = 4E^2. \]

Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation.

**This is a route to Peter Rowland’s Nilpotent operators.**
The “particle” $P$ interacts with $P$ to produce either $P$ or $*$. The particle $*$ is neutral.
Formally, we can distinguish the two interactions via adjacency and concentricity.
PP = * + P
\[ P^2 = * + P \]
\[ P^3 = P* + PP = P + * + P = * + 2P \]
\[ P^4 = P + 2* + 2P = 2* + 3P \]
\[ P^5 = 3* + 5P \]
\[ P^6 = 5* + 3P \]
\[ P^7 = 8* + 5P \]
The process space with three input P’s and one output P has dimension two. It is a candidate for a unitary representation of the three strand braids.
Fibonacci Tree of Particle Interactions
Fibonacci Tree:

Admissible Sequences are the Paths from the Root
A Formal Majorana Fermion

\[ P \cdot P \]

\[ \ast \quad P \]
An Elementary Particle has a Fusion Algebra and a Creation/Annihilation Algebra.

The Creation/Annihilation algebra for a Majorana Fermion is very simple. Just an element $a$ with $aa = 1$.

If there are two Majorana Fermions, we have $a, b$

with $aa = 1, bb = 1$ and $ab + ba = 0$. 
For a standard Fermion there is a
an annihilation operator $F$
and a creation operator $F^*$.

These correspond to the fact that
the antiparticle is distinct from the
particle.

We have $FF = F^*F^* = 0$
but
$FF^* + F^*F = 1$. 
Majorana Fermions are their own antiparticles.

Mathematically an Electron’s creation and annihilation operators are combinations of Majorana Fermion operators:

\[ U = a + ib \quad \text{and} \quad U^* = a - ib \]

where \( ab + ba = 0 \) and \( aa = bb = 1 \).

Are electrons composites of pairs of Majorana fermions?
Note \( UU = (a+ib)(a+ib) = aa - bb + i(ab + ba) = 0 \)
and \( U*U* = 0. \)

\[
UU* + U*U = (U + U*)(U+U*) = 4aa = 4
\]

This is (unnormalized) creation/annihilation algebra for an electron.
A possible sighting of Majorana states

Nearly 80 years ago, the Italian physicist Ettore Majorana proposed the existence of an unusual type of particle that is its own antiparticle, the so-called Majorana fermion. The search for a free Majorana fermion has so far been unsuccessful, but bound Majorana-like collective excitations may exist in certain exotic superconductors. Nadj-Perge et al. created such a topological superconductor by depositing iron atoms onto the surface of superconducting lead, forming atomic chains (see the Perspective by Lee). They then used a scanning tunneling microscope to observe enhanced conductance at the ends of these chains at zero energy, where theory predicts Majorana states should appear.
Unpaired Majorana fermions in quantum wires

Alexei Yu. Kitaev
Microsoft Research
Microsoft, #113/2032, One Microsoft Way,
Redmond, WA 98052, U.S.A.
kitaev@microsoft.com

27 October 2000

Figure 2: Two types of pairing.

Note that the state $|\psi_0\rangle$ has an even fermionic parity (i.e. it is a superposition of states with even number of electrons) while $|\psi_1\rangle$ has an odd parity. The parity is measured by the operator

$$P = \prod_j (-ic_{2j-1}c_{2j}).$$  \hspace{1cm} (9)
Non-abelian statistics of half-quantum vortices in $p$-wave superconductors

D. A. Ivanov

Institut für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
(May 11, 2000)

Excitation spectrum of a half-quantum vortex in a $p$-wave superconductor contains a zero-energy Majorana fermion. This results in a degeneracy of the ground state of the system of several vortices. From the properties of the solutions to Bogoliubov-de-Gennes equations in the vortex core we derive the non-abelian statistics of vortices identical to that for the Moore-Read (Pfaffian) quantum Hall state.

FIG. 3. Elementary braid interchange of two vortices.
Evidence for Majorana fermions in a nanowire

R. Mark Wilson

June 2012, page 14

DIGITAL OBJECT IDENTIFIER
http://dx.doi.org/10.1063/PT.3.1587

Electrical conductance measurements reveal what may be massless, chargeless, and spinless quasiparticles of zero energy.
**Majorana (real) Fermions**

\[ f^+, f \quad \text{Usual (complex) fermions} \]

\[ \psi = \frac{f^+ + f}{\sqrt{2}} \quad \psi = \psi^+ \quad \psi^2 = 1 \]

\[ f = \frac{\psi_1 + i\psi_2}{\sqrt{2}} \quad \text{“half” of the usual (complex) fermion} \]

\[ \text{“real” fermion} \]

\[ \psi_1, \psi_2 \quad \psi_3, \psi_4 \quad \psi_5, \psi_6 \quad \cdots \quad \psi_{2N-1}, \psi_{2N} \]

**Cooper pairing**

\[ \psi_1, \psi_{2N} \quad \text{Single fermion} \rightarrow 1 \text{ q-bit} \]
a. Pictorial representation of the ground state of equation (1) in the limit $\mu=0$, $t=|\Delta|$. Each spinless fermion in the chain is decomposed in terms of two Majorana fermions $\gamma_{A,i}$ and $\gamma_{B,i}$. Majoranas $\gamma_{B,i}$ and $\gamma_{A,i+1}$ combine to form an ordinary, finite-energy fermion, leaving two zero-energy end Majoranas $\gamma_{A,1}$ and $\gamma_{B,N}$ as shown.  

b. A spin–orbit-coupled semiconducting wire deposited on an s-wave superconductor can be driven into a topological superconducting state exhibiting such end Majorana modes by applying an external magnetic field. c. Band structure of the semiconducting wire when $B=0$ (dashed lines) and $B \neq 0$ (solid lines). When $\mu$ lies in the band gap generated by the field, pairing inherited from the proximate superconductor drives the wire into the topological state.
Let $a_1, a_2, a_3, \ldots, a_{2n}$ be Majorana Fermions.

\[
S(i) = \frac{(1 + a(i+1)a(i))}{\sqrt{2}} \\
S^\dagger(i) = \frac{(1 - a(i+1)a(i))}{\sqrt{2}}
\]

The operators $S(i)$ form a unitary representation of the Artin Braid Group:

\[
S(i)S(i+1)S(i) = S(i+1)S(i)S(i+1) \\
S(i)S(j) = S(j)S(i) \text{ when } |i-j|>2.
\]

Operators $T(i)$ act on the space of MF’s via
\[
T(i)x = S(i) \times S^\dagger(i) \\
T(i)a(i) = a(i+1) \\
T(i)a(i+1) = -a(i).
\]
Majoranas are related to standard fermions as follows: The algebra for Majoranas is \( c = c^\dagger \) and \( cc' = -c'c \) if \( c \) and \( c' \) are distinct Majorana fermions with \( c^2 = 1 \) and \( c'^2 = 1 \). One can make a standard fermion from two Majoranas via

\[
\psi = (c + ic')/2, \\
\psi^\dagger = (c - ic')/2.
\]

Similarly one can mathematically make two Majoranas from any single fermion. Now if you take a set of Majoranas

\[
\{ c_1, c_2, c_3, \cdots, c_n \}
\]

then there are natural braiding operators that act on the vector space with these \( c_k \) as the basis. The operators are mediated by algebra elements

\[
\tau_k = (1 + c_{k+1}c_k)/\sqrt{2}, \\
\tau_k^{-1} = (1 - c_{k+1}c_k)/\sqrt{2}.
\]

Then the braiding operators are

\[
T_k : \text{Span}\{ c_1, c_2, \cdots, c_n \} \longrightarrow \text{Span}\{ c_1, c_2, \cdots, c_n \}
\]

via

\[
T_k(x) = \tau_k x \tau_k^{-1}.
\]
It is worth noting that a triple of Majorana fermions say $a, b, c$ gives rise to a representation of the quaternion group. This is a generalization of the well-known association of Pauli matrices and quaternions. We have $a^2 = b^2 = c^2 = 1$ and they anticommute. Let $I = ba, J = cb, K = ac$. Then

$$I^2 = J^2 = K^2 = IJK = -1,$$

giving the quaternions. The operators

$$A = (1/\sqrt{2})(1 + I)$$
$$B = (1/\sqrt{2})(1 + J)$$
$$C = (1/\sqrt{2})(1 + K)$$

braid one another:

$$ABA = BAB, BCB = CBC, ACA = CAC.$$
Braiding Majorana Fermions

\[ T(x) = y \]
\[ T(y) = -x \]
And From Logic Alone?
Ludwig Wittgenstein

Tractatus Logico-Philosophicus

4.0621 That, however, the signs “p” and “\(\sim p\)” can say the same thing is important, for it shows that the sign “\(\sim\)” corresponds to nothing in reality.

5.511 How can the all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, to the great mirror.

5.6 The limits of my language mean the limits of my world.

5.632 The subject does not belong to the world but it is a limit of the world.
The Bare Bones of a Majorana Fermion from Logic Alone?

\[ \sim \sim Q = Q \text{ in Boolean logic.} \]

Can we write

\[ \sim \sim = \ast ? \]

Can negation interact with itself to produce Nothing (as above)?

Can negation interact with itself to produce itself?
In Laws of Form (G. Spencer-Brown)
Negation emerges from an operator that interacts with itself either to annihilate itself, or to produce itself.

\[
\begin{align*}
\overline{\overline{\overline{}} } &= * \quad \overline{\overline{\overline{}}} &= \overline{}
\end{align*}
\]

The Fibonacci particle is a “logical particle” for a level of logic deeper than Boolean Logic.
Interpretation for Logic

\[
\begin{align*}
\overline{A} & \iff \sim A \\
AB & \iff A \text{ or } B \\
* & \iff \text{True} \\
* & \iff \text{False}
\end{align*}
\]
Returning to Square Root of -1, Time and Dirac
Emergence of Fermion Algebra from Discrete Dynamical Process

So far we have seen that the mark can represent the fusion rules for a Majorana fermion since it can interact with itself to produce either itself or nothing. But we have not yet seen the anti-commuting fermion algebra emerge from this context of making a distinction. Remarkably, this algebra does emerge when one looks at the mark recursively.

Consider the transformation

\[ F(X) = \overline{X}. \]

If we iterate it and take the limit we find

\[ G = F(F(F(F(\cdots)))) = \overline{\overline{\overline{\overline{\cdots}}}} \]

an infinite nest of marks satisfying the equation

\[ G = \overline{G}. \]
THE SQUARE ROOT OF MINUS ONE IS A CLOCK.

From \( G = G \) to \( i = -1/i \).

\[ i \text{ as an imaginary value, defined in terms of itself.} \]

\[ i = -1/i \quad \text{ii} = -1 \]

The square root of minus one “is” a discrete oscillation.

\[ \ldots +1, -1, +1, -1, +1, -1, \ldots \]

\[ [-1, +1] \quad [+1, -1] \]
The Square Root of Minus One is a Clock.

From \( G = \overline{G} \) to \( i = -1/i \).

... +1, -1, +1, -1, +1, -1, +1, -1, ...

\[
T(x) = -1/x
\]

Fixed Point: \( i = -1/i \)
We introduce a *temporal shift operator* $\eta$ such that

$$[a, b] \eta = \eta [b, a]$$

and

$$\eta \eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\cdots abababab \cdots.$$  

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define $i$ by the equation

$$i = [1, -1] \eta.$$  

This makes $i$ both a value and an operator that takes into account a step in time. We calculate

$$ii = [1, -1] \eta [1, -1] \eta = [1, -1][-1, 1] \eta \eta = [-1, -1] = -1.$$
\[ e = [1, -1]. \]
\[ e^2 = [1, 1] = 1 \]
\[ [a, b] \eta = \eta [b, a] \quad i = [1, -1] \eta. \]
\[ e^2 = 1, \]
\[ \eta^2 = 1, \]
\[ e \eta = -\eta e. \]
\[ ii = [1, -1] \eta [1, -1] \eta = [1, -1] [-1, 1] \eta \eta = [-1, -1] = -1. \]
We can regard $e$ and $\eta$ as a fundamental pair of Majorana fermions. This is a formal correspondence, but it is striking how this Marjorana fermion algebra emerges from an analysis of the recursive nature of the reentering mark, while the fusion algebra for the Majorana fermion emerges from the distinctive properties of the mark itself. We see how the seeds of the fermion algebra live in this extended logical context.
Fermionic Spacetime

have the two Marjorana fermions $e$ and $\eta$ and the corresponding standard fermion creation and annihilation operators are then given by the formulas below.

$$\psi = (e + i\eta)/2$$

and

$$\psi^\dagger = (e - i\eta)/2.$$ 

Since $e$ represents a spatial view of the basic discrete oscillation and $\eta$ is the time-shift operator for this oscillation it is of interest to note that the standard fermion built by these two can be regarded as a quantum of spacetime, retrieved from the way that we decomposed the process into space and time. Since all this is initially built in relation to extending the Boolean logic of the mark to a non-boolean recursive context, there is further analysis needed of the relation of the
At this point we see that it is not just the square root of minus one that has emerged from the structure of the oscillation, but the simple non-commutative algebra of the split quaternions.

We will use this as an excuse to follow this algebra to its uses for Fermions and the Dirac equation.

In fact, we have uncovered a source of matrix algebra.
Let $A = [a,b]$ and $B = [c,d]$ and let $C = [r,s]$, $D = [t,u]$. With $A' = [b,a]$, we have

$$(A + B\eta)(C + D\eta) = (AC + BD') + (AD + BC')\eta.$$ 

This writes $2 \times 2$ matrix algebra in the form of a hypercomplex number system. From the point of view of iterants, the sum $[a,b] + [b,c]\eta$ can be regarded as a superposition of two types of observation of the iterants $I\{a,b\}$ and $I\{c,d\}$. The operator-view $[c,d]\eta$ includes the shift that will move the viewpoint from $[c,d]$ to $[d,c]$, while $[a,b]$ does not contain this shift. Thus a shift
5.2 Relativity and the Dirac Equation

Starting with the algebra structure of $e$ and $\eta$ and adding a commuting square root of $-1$, $i$, we have constructed fermion algebra and quaternion algebra. We can now go further and construct the Dirac equation. This may sound circular, in that the fermions arise from solving the Dirac equation, but in fact the algebra underlying this equation has the same properties as the creation and annihilation algebra for fermions, so it is by way of this algebra that we will come to the Dirac equation. If the speed of light is equal to 1 (by convention), then energy $E$, momentum $p$ and mass $m$ are related by the (Einstein) equation

$$E^2 = p^2 + m^2.$$
Dirac constructed his equation by looking for an algebraic square root of $p^2 + m^2$ so that he could have a linear operator for $E$ that would take the same role as the Hamiltonian in the Schrodinger equation. We will get to this operator by first taking the case where $p$ is a scalar (we use one dimension of space and one dimension of time. Let $E = \alpha p + \beta m$ where $\alpha$ and $\beta$ are elements of a a possibly non-commutative, associative algebra. Then

$$E^2 = \alpha^2 p^2 + \beta^2 m^2 + pm(\alpha \beta + \beta \alpha).$$

Hence we will satisfy $E^2 = p^2 + m^2$ if $\alpha^2 = \beta^2 = 1$ and $\alpha \beta + \beta \alpha = 0$. This is our familiar Clifford algebra pattern and we can use the iterant algebra generated by $e$ and $\eta$ if we wish. Then, because the quantum operator for momentum is $-i\partial/\partial x$ and the operator for energy is $i\partial/\partial t$, we have the Dirac equation

$$i\partial \psi / \partial t = -i\alpha \partial \psi / \partial x + \beta m \psi.$$

Let

$$\mathcal{O} = i\partial / \partial t + i\alpha \partial / \partial x - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O} \psi(x, t) = 0.$$
Now note that
\[ \mathcal{O} e^{i(px - Et)} = (E - \alpha p + \beta m) e^{i(px - Et)} \]
and that if
\[ U = (E - \alpha p + \beta m)\beta\alpha = \beta\alpha E + \beta p + \alpha m, \]
then
\[ U^2 = -E^2 + p^2 + m^2 = 0, \]
from which it follows that
\[ \psi = U e^{i(px - Et)} \]
is a (plane wave) solution to the Dirac equation.

In fact, this calculation suggests that we should multiply the operator \( \mathcal{O} \) by \( \beta\alpha \) on the right, obtaining the operator
\[ \mathcal{D} = \mathcal{O} \beta\alpha = i\beta\alpha \partial / \partial t + i\beta \partial / \partial x + \alpha m, \]
and the equivalent Dirac equation
\[ \mathcal{D}\psi = 0. \]

In fact for the specific \( \psi \) above we will now have \( \mathcal{D}(U e^{i(px - Et)}) = U^2 e^{i(px - Et)} = 0 \). This way of reconfiguring the Dirac equation in relation to nilpotent algebra elements \( U \) is due to Peter Rowlands [94]. We will explore this relationship with the Rowlands formulation in a separate paper.
Recapitulation

We start with $\psi = e^{i(px-Et)}$ and the operators

$$\hat{E} = i\partial/\partial_t$$

and

$$\hat{p} = -i\partial/\partial_x$$

so that

$$\hat{E}\psi = E\psi$$

and

$$\hat{p}\psi = p\psi.$$  

The Dirac operator is

$$\mathcal{O} = \hat{E} - \alpha\hat{p} - \beta m$$

and the modified Dirac operator is

$$\mathcal{D} = \mathcal{O}\beta\alpha = \beta\alpha\hat{E} + \beta\hat{p} - \alpha m,$$

so that

$$\mathcal{D}\psi = (\beta\alpha E + \beta p - \alpha m)\psi = U\psi.$$
\[ \mathcal{D}(U e^{i(px - Et)}) = U^2 e^{i(px - Et)} = 0. \]
If we let
\[ \tilde{\psi} = e^{i(px + Et)} \]
(reversing time), then we have
\[ D\tilde{\psi} = (-\beta \alpha E + \beta p - \alpha m)\psi = U^\dagger \tilde{\psi}, \]
giving a definition of \( U^\dagger \) corresponding to the anti-particle for \( U\psi \).

We have that
\[ U^2 = U^\dagger 2 = 0 \]
and
\[ UU^\dagger + U^\dagger U = 4E^2. \]

Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation.
Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation. Furthermore, the decomposition of $U$ and $U^\dagger$ into the corresponding Majorana Fermion operators corresponds to $E^2 = p^2 + m^2$. Normalizing by dividing by $2E$ we have

$$A = (\beta p \alpha m)/E$$

and

$$B = i\beta \alpha.$$ 

so that

$$A^2 = B^2 = 1$$

and

$$AB + BA = 0.$$ 

then

$$U = (A + Bi)E$$ 

and

$$U^\dagger = (A - Bi)E,$$

showing how the Fermion operators are expressed in terms of the simpler Clifford algebra of Majorana operators (split quaternions once again).
Note that we get different decompositions of the Fermion into Majorana operators according to what is reversed.

\{E\} and \{p,m\} for time reversal.
\{p\} and \{E,m\} for spin reversal.
\{E,p\} and \{m\} for spin and time reversal.
Writing in the Full Dirac Algebra

We have written the Dirac equation so far in one dimension of space and one dimension of time. We give here a way to boost the formalism directly to three dimensions of space. We take an independent Clifford algebra generated by $\sigma_1, \sigma_2, \sigma_3$ with $\sigma_i^2 = 1$ for $i = 1, 2, 3$ and $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$. Now assume that $\alpha$ and $\beta$ as we have used them above generate an independent Clifford algebra that commutes with the algebra of the $\sigma_i$. Replace the scalar momentum $p$ by a 3-vector momentum $p = (p_1, p_2, p_3)$ and let $p \cdot \sigma = p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3$. We replace $\partial / \partial x$ with $\nabla = (\partial / \partial x_1, \partial / \partial x_2, \partial / \partial x_2)$ and $\partial p / \partial x$ with $\nabla \cdot p$.

We then have the following form of the Dirac equation.

$$i \partial \psi / \partial t = -i \alpha \nabla \cdot \sigma \psi + \beta m \psi.$$ 

Let

$$\mathcal{O} = i \partial / \partial t + i \alpha \nabla \cdot \sigma - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O} \psi(x, t) = 0.$$
In analogy to our previous discussion we let

\[ \psi(x, t) = e^{i(p \cdot x - Et)} \]

and construct solutions by first applying the Dirac operator to this \( \psi \). The two Clifford algebras interact to generalize directly the nilpotent solutions and Fermion algebra that we have detailed for one spatial dimension to this three dimensional case. To this purpose the modified Dirac operator is

\[ D = i \beta \alpha \partial / \partial t + \beta \nabla \cdot \sigma - \alpha m. \]

And we have that

\[ D \psi = U \psi \]

where

\[ U = \beta \alpha E + \beta p \cdot \sigma - \alpha m. \]

We have that \( U^2 = 0 \) and \( U \psi \) is a solution to the modified Dirac Equation, just as before. And just as before, we can articulate the structure of the Fermion operators and locate the corresponding Majorana Fermion operators. We leave these details to the reader.
The Majorana-Dirac Equation

There is more to do. We will end with a brief discussion making Dirac algebra distinct from the one generated by $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3$ to obtain an equation that can have real solutions. This was the strategy that Majorana [7] followed to construct his Majorana Fermions. A real equation can have solutions that are invariant under complex conjugation and so can correspond to particles that are their own anti-particles. We will describe this Majorana algebra in terms of the split quaternions $\epsilon$ and $\eta$. For convenience we use the matrix representation given below. The reader of this paper can substitute the corresponding iterants.

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Let $\hat{\epsilon}$ and $\hat{\eta}$ generate another, independent algebra of split quaternions, commuting with the first algebra generated by $\epsilon$ and $\eta$. Then a totally real Majorana Dirac equation can be written as follows:

$$(\partial/\partial t + \hat{\eta} \partial/\partial x + \epsilon \partial/\partial y + \hat{\epsilon} \partial/\partial z - \epsilon \eta m)\psi = 0.$$  

To see that this is a correct Dirac equation, note that

$$\hat{E} = \alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m$$

(Here the “hats” denote the quantum differential operators corresponding to the energy and momentum.) will satisfy

$$\hat{E}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + m^2$$
if the algebra generated by $\alpha_x, \alpha_y, \alpha_z, \beta$ has each generator of square one and each distinct pair of generators anti-commuting. From there we obtain the general Dirac equation by replacing $\hat{E}$ by $i\partial/\partial t$, and $\hat{p}_x$ with $-i\partial/\partial x$ (and same for $y, z$).

$$(i\partial/\partial t + i\alpha_x \partial/\partial x + i\alpha_y \partial/\partial y + i\alpha_z \partial/\partial y - \beta m)\psi = 0.$$ 

This is equivalent to

$$(\partial/\partial t + \alpha_x \partial/\partial x + \alpha_y \partial/\partial y + \alpha_z \partial/\partial y + i\beta m)\psi = 0.$$ 

Thus, here we take

$$\alpha_x = \hat{\eta}, \alpha_y = \epsilon, \alpha_z = \hat{\epsilon} \eta, \beta = i\hat{\epsilon} \eta;$$

and observe that these elements satisfy the requirements for the Dirac algebra. Note how we have a significant interaction between the commuting square root of minus one ($i$) and the element $\hat{\epsilon} \eta$ of square minus one in the split quaternions. This brings us back to our original considerations about the source of the square root of minus one. Both viewpoints combine in the element $\beta = i\hat{\epsilon} \eta$ that makes this Majorana algebra work. Since the algebra appearing in the Majorana Dirac operator is constructed entirely from two commuting copies of the split quaternions, there is no appearance of the complex numbers, and when written out in $2 \times 2$ matrices we obtain coupled real differential equations to be solved. Clearly this ending is actually a beginning of a new study of Majorana Fermions. That will begin in a sequel to the present paper.
The Feynman Checkerboard

\[ \psi = \sum_P i^C(P) \]

Dirac Amplitude

\[ C(P) = \text{number of corners in path } P \]

\[ r = (t+x)/2 \]
\[ \ell = (t-x)/2 \]
Dirac Equation and Feynman Checkerboard

More specifically, let \((a, b)\) denote a point in discrete Minkowski spacetime in lightcone coordinates. This means that \(a\) denotes the number of steps taken to the left and \(b\) denotes the number of steps taken to the right. We let \(\psi_L(a, b)\) denote the sum over the paths that enter the point \((a, b)\) from the left and \(\psi_R(a, b)\) the sum over the paths that enter \((a, b)\) from the right. Each path \(P\) contributes \(i^{c(P)}\) where \(c(P)\) denotes the number of corners in the path. View the diagram below.

\[
\begin{align*}
(a, b+1) & \quad \rightarrow \\
(a, b) & \quad \rightarrow \quad (a, b+1)
\end{align*}
\]

It is clear from the diagram that

\[
\psi_L(a, b+1) = \psi_L(a, b) + i\psi_R(a, b).
\]

Thus we have that

\[
\frac{\partial \psi_L}{\partial R} = i\psi_R
\]

and similarly

\[
\frac{\partial \psi_R}{\partial L} = i\psi_L.
\]

This pair of equations is the Dirac equation in light cone coordinates.
To Get a Real Valued Equation:

\[ i \frac{\partial \psi}{\partial t} = \left( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) \psi \]

\[ \frac{\partial \psi}{\partial t} = \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \psi . \]
If $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ where $\psi_1$ and $\psi_2$ are real-valued functions of $x$ and $t$, then we have

\[
\begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \psi_1}{\partial t} - \frac{\partial \psi_1}{\partial x} \\ \frac{\partial \psi_2}{\partial t} + \frac{\partial \psi_2}{\partial x} \end{pmatrix}.
\] (9)

Now the light cone coordinates of a point $(x, t)$ of space-time are given by $[r, \ell] = \left[ \frac{1}{2}(t + x), \frac{1}{2}(t - x) \right]$ and hence the Dirac equation becomes

\[
\begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \psi_1}{\partial \ell} \\ \frac{\partial \psi_2}{\partial r} \end{pmatrix}.
\] (10)
In the RII, Majorana Fermion case we have

\[
\frac{\partial \psi_2}{\partial r} = \psi_2
\]

\[
\frac{\partial \psi_1}{\partial l} = -\psi_1
\]

Thus the Checkerboard works with plus/minus cornering. This model gives a picture of how Majorana’s equations can look in an explicit case and show how discrete quantum physics may avoid complex numbers.
But we should take a wider viewpoint. The universal equation should be about the (state of) the Universe $U$.

An operator $D$ acts on $U$ to produce Nothing.

$$D \ U = 0.$$  

But the universe $U$ is both operator and operand. So we take $D = U$ and obtain the Universal Nilpotent equation.

$$UU = 0,$$

of which the Dirac equation is one of the first special cases.
The Simplest example of the Universal Nilpotent Equation is given by the operator

\[ Ux = x \]

Here the Universe \( U \) is that Universe (self) created by the Mark and taken to Nothing by the crossing from the marked state to the unmarked state.

\[ UU = x = 0. \]

\[ UU = 0. \]
In this formalism the mark is seen to make a distinction (in the plane). The formal language of the calculus of indications refers to the mark and is built from the mark.

The language using the mark is inherently self-referential. The mark and the observer are seen, in the form, to be identical. The Calculus writes itself in terms of itself.
Physical theory is seen to write itself in terms of the condition for observation to occur at all.
Discrete Physics and Non-Commutative Frameworks
Discrete Measurement is Intrinsically Non-commutative.

Time Series: $X, X', X'', \ldots$

Derivative: $X = (X' - X)/dt = dX/dt$

Here $dt$ and $dX$ are finite increments.

$XX$: Observe $X$, then observe $X$.

$XX$: Observe $X$, then observe $X$.

$XX = X'(X' - X)/dt$

$XX = (X' - X)X/dt$

$XX - XX = (X' - X)(X' - X)/dt$

$[X, X] = (dX)^2/dt$
\[ XX - \dot{XX} = (X' - X)(X' - X)/dt \]

\[ [X, X] = K \text{ then } K = (dx)(dx)/dt \]

\[ X' = X \pm dx \]

The discrete analog of Heisenberg's equation yields a Brownian walk with diffusion constant \( K \).
Recalling the Diffusion Constant and the Diffusion Equation

\[ P(x,t) = \text{Probability that the particle is at } x \text{ at time } t. \]

\[ P(x,t+dt) = \frac{1}{2}[P(x+dx,t) + P(x-dx,t)] , \text{whence} \]

\[ P(x,t+dt) - P(x,t) = \frac{1}{2}[P(x+dx,t) - P(x,t) -(P(x,t) - P(x-dx,t))] \]

\[ \frac{dP}{dt} = \frac{K}{2} \frac{d^2 P}{dx^2} \text{ Diffusion Equation} \]

\[ K = (dx)^2 / dt \text{ Diffusion Constant} \]

We have just seen the diffusion constant arise differently(!) in the context of discrete process commutators, with no second difference.
Discrete calculus is embedded in commutator calculus:

\[ \dot{X} \]

is a signal to time-shift the algebra to its left.

Make algebraic by defining new operator $J$ with

\[ XJ = JX'. \]

Redefine

\[ \dot{X} = J(X' - X)/dt. \]

Then

\[ \dot{X} = (XJ - JX)/dt = [X, J/dt]. \]

\[ \dot{X} \]
satisfies the Leibniz rule.
A non-commutative world of flat coordinates suitable for advanced calculus.

The flat coordinates $X_i$ satisfy the equations below with the $P_j$ chosen to represent differentiation with respect to $X_j$:

$$[X_i, X_j] = 0, \quad \text{Coordinates Commute.}$$
$$[P_i, P_j] = 0, \quad \text{Partials commute.}$$
$$[X_i, P_j] = \delta_{ij}. \quad \text{Derivative formula.}$$

Derivatives are represented by commutators.

$$\partial_i F = \partial F / \partial X_i = [F, P_i],$$
$$\hat{\partial}_i F = \partial F / \partial P_i = [X_i, F].$$

Temporal derivative is represented by commutation with a special (Hamiltonian) element $H$ of the algebra:

$$dF/dt = [F, H].$$

(For quantum mechanics, take $i\hbar dA/dt = [A, H]$.)

Hamilton’s Equations express the Mathematics of a Non-Commutative Flat World.

\[
dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H/\partial X_i
\]

\[
dX_i/dt = [X_i, H] = \partial H/\partial P_i.
\]

These are exactly Hamilton’s equations of motion. The pattern of Hamilton’s equations is built into the system.
General Equations of Motion

Dynamics and Gauge Theory. One can take the general dynamical equation in the form

\[ \frac{dX_i}{dt} = G_i \]

where \( \{G_1, \cdots, G_d\} \) is a collection of elements of \( A \). Write \( G_i \) relative to the flat coordinates via \( G_i = P_i - A_i \).

\[ R_{ij} = [G_i, G_j] \]

\[ = [P_i - A_i, P_j - A_j] \]

\[ = -[P_i, A_j] - [A_i, P_j] + [A_i, A_j] \]

\[ = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \]

This is the well-known formula for the curvature of a gauge connection.
One can explore formulations of discrete physics in a non-commutative context.

The context of non-commutative calculus where the derivatives are represented by commutators is directly related to physics in a new way by this translation of the discrete. This also suggests opening the books again on the relationship of commutators and quantum theory.
Knots, Links and Lie Algebras
Vassiliev Invariants

\( v(K^*) = v(K^+) - v(K^-) \)

Skein Identity

Chord Diagram
Four-Term Relation From Topology

\[
\begin{array}{c}
\xrightarrow{E} - \xrightarrow{E} = \xrightarrow{E} \\
\xleftarrow{E} - \xleftarrow{E} = -\xleftarrow{E} \\
\xrightarrow{E} - \xleftarrow{E} = -\xrightarrow{E} \\
\xleftarrow{E} - \xrightarrow{E} = -\xleftarrow{E} \\
\xrightarrow{E} - \xrightarrow{E} + \xleftarrow{E} = 0
\end{array}
\]
Four Term Relation from Lie Algebra

\[ \tau^a \tau^b - \tau^b \tau^a = f^c_{ab} \tau^c \]
The Jacobi Identity

\[(a \bullet b) \bullet c - (a \bullet c) \bullet b = a \bullet (b \bullet c)\]

Hence
\[(a \bullet b) \bullet c + b \bullet (a \bullet c) = a \bullet (b \bullet c).\]
Rishon Model of Elementary Particles

by J. D. Shelton

It has been proposed that the quarks and leptons consist of more fundamental particles called rishons. The T rishon may be defined as having mass and charge $e/3$. The V rishon is neutral and has little or no mass. The rishons have spin $\frac{1}{2}$, carry color charge, and combine in triplets or rishon-antirishon pairs. Thus the electron is a TTT, the neutrino VVV, the down quark TVV, and the up quark $TTV$. If the T has somewhat greater color charge than the V, the down quark would have a net excess of the color carried by the T. The antiup quark $TTV$ would appear to have a net deficiency of the color carried by the V, or equivalently, an excess of anticolor, and behave as an antiparticle. Hence the $TTV$ would appear to have an excess of color and behave as a particle, in agreement with observation. The leptons have no net color. There is no need for hypercolor.

All particle interactions consist of rearrangements of rishons, or creation or annihilation of rishon-antirishon pairs. For example, beta-decay occurs when a down quark changes to an up quark, emitting an electron and neutrino:

$$TVV \rightarrow TTV + TTT + VVV$$

The massless particle was originally called a neutrino; it was later defined to be an antineutrino. This model favors the first choice.
TTT positron
TTV up quark, red
TVT up quark, green
VTT up quark, blue
VVT anti down quark, red
VTV anti down quark, green
TVV anti down quark, blue
VVV electron neutrino
A topological model of composite preons

Sundance O. Bilson-Thompson*

Centre for the Subatomic Structure of Matter, Department of Physics,
University of Adelaide, Adelaide SA 5005, Australia

(Dated: February 2, 2008)

We describe a simple model, based on the preon model of Shupe and Harari, in which the binding of preons is represented topologically. We then demonstrate a direct correspondence between this model and much of the known phenomenology of the Standard Model. In particular we identify the substructure of quarks, leptons and gauge bosons with elements of the braid group $B_3$. Importantly, the preonic objects of this model require fewer assumed properties than in the Shupe/Harari model, yet more emergent quantities, such as helicity, hypercharge, and so on, are found. Simple topological processes are identified with electroweak interactions and conservation laws. The objects which play the role of preons in this model may occur as topological structures in a more comprehensive theory, and may themselves be viewed as composite, being formed of truly fundamental sub-components, representing exactly two levels of substructure within quarks and leptons.

PACS numbers: 12.60.Rc, 12.10.Dm
II. THE HELON MODEL

Let us now introduce our topologically-based toy model of quarks, leptons, and gauge bosons. It is convenient to represent the most fundamental objects in this model by twists through $\pm \pi$ in a ribbon. For convenience let us denote a twist through $\pi$ as a “dum”, and a twist through $-\pi$ as a “dee” ($U$ and $E$ for short, after Tweedle-dum and Tweedle-dee [7]). Generically we refer to such twists by the somewhat whimsical name “tweedles”[8]. We hope to deduce the properties of quarks and leptons and their interactions from the behaviour of their constituent tweedles, and to do so we shall employ a set of assumptions that govern their behaviour:
1) **Unordered pairing:** Tweedles combine in pairs, so that their total twist is 0 modulo $2\pi$, and the ordering of tweedles within a pair is unimportant. The three possible combinations of $UU$, $EE$, and $UE \equiv EU$ can be represented as ribbons bearing twists through the angles $+2\pi$, $-2\pi$, and 0 respectively. A twist through $\pm 2\pi$ is interpreted as an electric charge of $\pm e/3$. We shall refer to such pairs of tweedles as helons (evoking their helical structure) and denote the three types of helons by $H_+$, $H_-$, and $H_0$.

2) **Helons bind into triplets:** Helons are bound into triplets by a mechanism which we represent as the tops of each strand being connected to each other, and the bottoms of each strand being similarly connected. A triplet of helons may split in half, in which case a new connection forms at the top or bottom of each resulting triplet. The reverse process may also occur when two triplets merge to form one triplet, in which case the connection at the top of one triplet and the bottom of the other triplet “annihilate” each other.

The arrangement of three helons joined at the top and bottom is equivalent to two parallel disks connected by a triplet of strands. In the simplest case, such an arrangement is invariant under rotations through angles of $2\pi/3$, making it impossible to distinguish the strands without arbitrarily labelling or colouring them. However we can envisage the three strands crossing over or under each other to form a braid. The three strands can then be distinguished by their relative crossings. We will argue below that braided triplets represent fermions, while unbraided triplets provide the simplest way to represent gauge bosons.

3) **No charge mixing:** When constructing braided triplets, we will not allow $H_+$ and $H_-$ helons in the same triplet. $H_+$ and $H_0$ mixing, and $H_-$ and $H_0$ mixing are allowed.

4) **Integer charge:** All unbraided triplets must carry integer electric charge.
FIG. 2: A representation of the decay $\mu \rightarrow \nu_\mu + e^- + \bar{\nu}_e$, showing how the substructure of fermions and bosons demands that charged leptons decay to neutrinos of the same generation.
Figure 7: Neutron Decay Model

**Avrin**

**Bilson-Thompson**

**Comparing Topological Formalisms**
FIG. 1: The fermions formed by adding zero, one, two or three charges to a neutral braid. Charged fermions come in two handedness states each, while $\nu$ and $\bar{\nu}$ come in only one each. (3) denotes that there are three possible permutations, identified as the quark colours. The bands at top and bottom represent the binding of helons.
FIG. 3: The bosons of the electroweak interaction. Notice that the $Z^0$ and the photon can deform into each other.
The Topology/Stability of the Fabric

It is interesting to note that three helons seems to be the minimum number from which a stable, non-trivial structure can be formed. By stable, we mean that a physical representation of a braid on three strands (e.g. made from strips of fabric) cannot in general be smoothly deformed into a simpler structure. By contrast, such a physical model with only two strands can always be un-twisted.

Can we imagine these “particles” as bits of surface interacting in a “superficial spin foam”.

Bilson-Thompson, Sundance (3-PITP); Hackett, Jonathan (3-PITP); Kauffman, Louis H. (1-ILCC-MS)

Particle topology, braids, and braided belts. (English summary)

Basic Twists
Twist 1

Twist 2

\begin{align*}
    \frac{1}{2} & \quad \frac{1}{2} \\
    \frac{-1}{2} & \quad \frac{-1}{2} \\
    0 & \quad 0 \\
    \end{align*}

\text{Figu 2:} The Twist

\text{Fig 3:} The Twist on a closed 3-Belt

\approx
Twist Concatenation Makes Braided Belts
Begin by cutting two slits into a strip of leather.
Be careful not to cut all the way to the ends.

Holding the top flat, pull string C over string B, and pull string A over string C.

Next, pull string B over string A, and pull string C over string B.

Now pull string A over string C, and pull string B over string A.

Untangle the bottom portion by sliding the bottom end through the open slits.

Continue this pattern until the braid reaches the bottom of the strip.
\[
\begin{align*}
\sigma_1 &= [-1/2, -1/2, +1/2] P_{12} \\
\sigma_1^{-1} &= [+1/2, +1/2, -1/2] P_{12} \\
\sigma_2 &= [+1/2, -1/2, -1/2] P_{23} \\
\sigma_2^{-1} &= [-1/2, +1/2, +1/2] P_{23}
\end{align*}
\]
\[ \sigma_2 \sigma_1^{-1} = [+1/2, -1/2, -1/2] P_{23} [+1/2, +1/2, -1/2] P_{12} \]
\[ = [+1/2, -1/2, -1/2][+1/2, -1/2, +1/2] P_{23} P_{12} \]
\[ = [+1, -1, 0] P_{23} P_{12} \]
\[ = [+1, -1, 0] P_{(132)} \]
\[(\sigma_2\sigma_1^{-1})^3 = \]
\[= [+1, -1, 0]P_{(132)}[+1, -1, 0]P_{(132)}[+1, -1, 0]P_{(132)} \]
\[= [+1, -1, 0]P_{(132)}[+1, -1, 0][-1, 0, +1]P_{(132)}P_{(132)} \]
\[= [+1, -1, 0]P_{(132)}[0, -1, +1]P_{(132)}P_{(132)} \]
\[= [+1, -1, 0][-1, +1, 0]P_{(132)}P_{(132)}P_{(132)} \]
\[= [0, 0, 0]I. \]
The Positron and Its Linked Boundary

\[ e^+ = [1, 1, 1] \sigma_1^{-1} \sigma_2 = [1, 1, 1][1/2, 1/2, -1/2]P_{12}[1/2, -1/2, -1/2]P_{23} \]

\[ = [1, 1, 1][1/2, 1/2, -1/2][-1/2, 1/2, -1/2]P_{12}P_{23} = [1, 2, 0]P_{12}P_{23}. \]
Apparent Moral:
There is topological persistence in particle properties for the surfaces.

To what extent do the surfaces represent the elementary particles? To what extent does the framed braid and/or its algebraic representation represent the elementary particles?
A permutation matrix $P$ is a matrix whose columns are a permutation of the rows of the identity matrix. Such a matrix acts as a permutation on the standard basis (column vectors that have a single unit entry). We shall refer to $P$ as a permutation. If $D$ is a diagonal matrix and $P$ is a permutation matrix, then

$$PD = D^P P$$

where $D^P$ denotes the result of permuting the elements of $D$ along the diagonal according to the permutation $P$. 
Let \([a, b, c]\) denote the matrix

\[
[a, b, c] = \begin{pmatrix}
t^a & 0 & 0 \\
0 & t^b & 0 \\
0 & 0 & t^c \\
\end{pmatrix}.
\]

Let

\[
P_{12} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

and

\[
P_{23} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}.
\]
This is the (framed) braid group representation with which we have been working.